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Does Noncommutative Geometry Predict Nonlinear Higgs Mechanism?

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It is argued that the noncommutative geometry construction of the standard model may predict a nonlinear symmetry-breaking mechanism rather than the orthodox Higgs mechanism if there is much heavy generation in addition to the lightest generations. Such models have experimentally verifiable consequences.

The unification of electromagnetic and weak interactions is one of the biggest achievements of theoretical physics. It is usually referred to as the Glashow–Weinberg–Salam model (GSW model). This model successfully described all known experiments involving electroweak interactions. We believe that the existence of the Higgs particle and the missing members of the third family will soon be confirmed. The situation is far less satisfactory from the theoretical point of view: the model contains too many free parameters and the symmetry-breaking sector is put in ad hoc. String theory (Sładkowski, 1990*a*) may provide us with an explanation for the nature of (light) generations (Mańka and Sładkowski, 1989, 1990; Sładkowski, 1990*b*). Recently, new ideas have been put forward (Connes, 1990; Connes and Lott, 1990; Kastler, 1991, 1992; Coquereau *et al.*, 1991, 1992; Chamsedine *et al.*, 1993*a*; Várilly and Garcia-Bondía, 1993) that make use of Connes' noncommutative geometry (Connes, 1983). Connes managed to reformulate the standard notions of differential geometry in a pure algebraic way that allows one to get rid of continuity and differentiability. As there is a geometrical interpretation of gauge theory in terms of fiber bundles and connections on them, one can also apply this formalism to the

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GSW model (Connes, 1990; Connes and Lott, 1990; Kastler, 1991, 1992; Coquereau *et al.*, 1991, 1992; Chamsedine *et al.*, 1992; Várilly and Garcia-Bondía, 1993) and to grand unification (Chamsedine *et al.*, 1993*a*). The notion of spacetime manifold M described by the (commutative) algebra of functions on M can be generalized to (*a priori*) an arbitrary noncommutative algebra. Fiber bundles become projective modules. A properly generalized connection can describe gauge field on these algebraic structures. The reader is referred to Connes (1983, 1990), Connes and Lott (1990), Kastler (1991, 1992), Coquereau *et al.* (1991, 1992), Chamsedine *et al.* (1993*a*), and Várilly and Garcia-Bondía (1993) for details. This allows one to incorporate the Higgs field into the gauge field, and the correct (leading to spontaneous symmetry-breaking) form of the scalar potential is obtained in a natural way, provided there are at least two generations of fermions! This sort of unification determines also the (classical) value of the Weinberg angle. One can add QCD to the model in such a way that the full standard model is reproduced. The Lagrange function one gets has the orthodox form with the above predictions. Of course, these predictions may get renormalized after quantization. Toy models suggest that it is difficult to keep the relations intact. Probably one should invent a noncommutative generalization of quantization in order to exploit the noncommutative character of the approach.

Here we would like to point out that the noncommutative generalization may predict a nonlinearly realized spontaneous symmetry breaking, known under the acronym BESS (breaking electroweak sector strongly) (Cosalbuoni *et al.*, 1987; Cvetic and Kögerler, 1989, 1991; Bönish and Kögerler, 1992). Our main argument for BESS can be stated as follows. The noncommutative version of the standard model predicts the required form of the Higgs sector, but the fermion masses (Yukawa couplings) and the number of generations N_g are free parameters. There must be at least two generations, but why not, say, 127? It is natural to suppose that N_g is big or even unlimited and that the fermion masses emerge as a result of interaction and the spacetime structure. We see only the lightest fermions because the energy at our disposal is not high enough. The Higgs particle has not yet been discovered. Does it really exist as a physical particle? We will show that it can be thought of in the limit $m_H \to \infty$. The main argument against BESS is that such models are nonrenormalizable. Noncommutative geometry says that our notion of spacetime is only an approximation (an effective electromagnetic spacetime). The correct description is in terms of algebras. Should we not give up the requirement of renormalizability? BESS models can certainly lead to physical predictions (Cvetic and Kögerler, 1989, 1991; Bönish and Kögerler, 1992). General relativity provides us with analogous arguments (Chamsedine *et al.*, 1993*b*).

We will consider a noncommutative space (A, h, D, Γ) where A is an involutive algebra, h a Hilbert space, D an unbounded self-adjoint operator on h (Dirac operator), and Γ a grading such that A is even and D is odd. Γ will provide us with the γ_5 matrix (Connes, 1983, 1990; Connes and Lott, 1990; Kastler, 1991, 1992; Coquereau *et al.*, 1991, 1992; Chamsedine *et al.*, 1992*a*; Várilly and Garcia-Bondía, 1993). We shall choose A to be the algebra that corresponds to the two-point extension of the spacetime (Connes, 1990; Connes and Lott, 1990; Kastler, 1991, 1992; Coquereau *et al.*, 1991, 1992; Chamsedine *et al.*, 1992*a*, 1993*b*; Várilly and Garcia-Bondía, 1993):

$$A = C^{\infty}(M) \otimes A_2 \tag{1}$$

where $C^{\infty}(M)$ is the algebra of functions on the spacetime (spin) manifold M and A_2 is the direct sum

$$A_2 = \mathbf{H} \oplus \mathbf{C} \tag{2}$$

of quarternions **H** and complex numbers **C**. The Hilbert space h has the form

$$L^2(S(M)) \otimes \mathbf{C}^{N_g} \tag{3}$$

where $L^2(S(M))$ denotes the Hilbert space of the square-integrable spinors [completion of the sections of the spin bundle $S(M)$]. The total fermion space has the form

$$L^2(S(M)) \otimes \mathbf{C}^{7N_g} \tag{4}$$

This corresponds to the fermions written in the form

$$\psi = \begin{pmatrix} \nu_L \\ e_L \\ u_L \\ d_L \\ e_R \\ u_R \\ d_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{5}$$

where each entry describes N_g ordinary fermions. So far, we have only considered the electroweak sector of the standard model. The QCD part should be added "in a commutative way" because the $SU(3)_{\text{color}}$ is an unbroken symmetry (Connes, 1990; Connes and Lott, 1990; Kastler, 1991, 1992; Coquereau *et al.*, 1991, 1992; Várilly and Garcia-Bondía, 1993). To this end one has to consider the algebra

$$\tilde{A} = C^{\infty}(M) \otimes A_c = C^{\infty}(M) \otimes A_2 \otimes (\mathbf{C} \oplus \mathbf{C}^{3\times 3}) \tag{6}$$

The additional **C** term introduces an extra $U(1)$ symmetry, the price one has to pay for having $SU(3)_{\text{color}}$ symmetry (no appropriate Higgs fields). The symmetry is now $U(1)_1 \times SU(2) \times U(1)_2 \times U(3)$. To reduce it to the standard one, one has to demand that the $U(1)_1$ part of the associated connection Y is equal to the trace part of the $U(3)$ term and that the $U(1)_2$ part is equal to $-Y$ (Connes and Lott, 1990). A more elegant but equivalent treatment can be found in Várilly and Garcia-Bondía (1993). This defines the algebraic structure of the full standard model. The Dirac operator has the form

$$D = \begin{pmatrix} \partial\!\!\!/ \otimes \mathrm{Id} & \gamma_5 \otimes M^\dagger \\ \gamma_5 \otimes M & \partial\!\!\!/ \otimes \mathrm{Id} \end{pmatrix} \tag{7}$$

where

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_d \\ 0 & m_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_d^\dagger & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

and the entries m_e, m_u and m_d are positive-definite $N_g \times N_g$ matrices. The Yang–Mills functional is defined by a representation π: $\Omega^*(A) \to B(h)$ of the differential algebra $\Omega^*(A)$ in the Hilbert space h in terms of bounded operators on h

$$\pi(a_0\, da_1 \ldots da_k) = a_0 i^k [D, a_1] \ldots [D, a_k] \tag{9}$$

by

$$L_{\text{YM}} = \frac{1}{4} \mathrm{Tr}_\omega((\pi^2(\theta)) D^{-4}) = \frac{1}{4} \int d^4x\, \mathrm{Tr}(\mathrm{tr}(\pi^2(\theta))) \tag{10}$$

where θ is the noncommutative curvature form, $\theta = d\rho + \rho^2$. Here Tr_ω, Tr, and tr denote the Diximier trace, trace over the matrices, and trace over the Clifford algebra, respectively (Connes, 1983, 1990; Connes and Lott, 1990; Várilly and Garcia-Bondía, 1993). We have

$$\rho = \begin{bmatrix} \tilde{A}_1 \otimes \mathrm{Id} & \gamma_5 \otimes H & 0 & 0 \\ \gamma_5 \otimes H^\dagger & \tilde{A}_2 \otimes \mathrm{Id} & 0 & 0 \\ 0 & 0 & -\tilde{A}_2 \otimes \mathrm{Id} & 0 \\ 0 & 0 & 0 & \tilde{A}_{\text{color}} \otimes \mathrm{Id} \end{bmatrix} \tag{11}$$

$$\tilde{A}_1 = \begin{pmatrix} \tilde{A}^3 & \tilde{A}^1 - i\tilde{A}^2 \\ \tilde{A}^1 + i\tilde{A}^2 & -\tilde{A}^3 \end{pmatrix} \tag{12}$$

$$\tilde{A}_2 = i\tilde{A}^0 = \mathrm{Tr}\, \tilde{A}_{\mathrm{color}} \tag{13}$$

and $W^+ = (1/\sqrt{2})(\tilde{A}^1 - i\tilde{A}^2)$, $Z = (1/\sqrt{2})(\tilde{A}^0 + \tilde{A}^3)$, etc. The tilde is used to denote gauge fields of the corresponding algebras. After elimination of auxiliary fields and Wick-rotating to Minkowski space, we get

$$\begin{aligned} L_{\mathrm{YM}} = \int \{ & \tfrac{1}{4} N_g (F^1_{\mu\nu} F^{1\mu\nu} + F^2_{\mu\nu} F^{2\mu\nu} + F^c_{\mu\nu} F^{c\mu\nu}) \\ & + \tfrac{1}{2} \mathrm{Tr}(MM^\dagger) |\partial H + A_1 H - H^\dagger A_2|^2 \\ & - \tfrac{1}{2} (\mathrm{Tr}(MM^\dagger)^2 - (\mathrm{Tr}\, MM^\dagger)^2)(HH^\dagger - 1)^2 \}\, d^4x \end{aligned} \tag{14}$$

The fermionic action is given by

$$\begin{aligned} L_f &= \langle \psi | D + \pi(\rho) | \psi \rangle \\ &= \int (\bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R + \bar{\psi}_L H \otimes M \psi_R + \bar{\psi}_R H^\dagger \otimes M^\dagger \psi_L)\, d^4x \end{aligned} \tag{15}$$

where we have included the $\pi(\rho)$ term in $\not{D}$.

Let us look more closely at the full Lagrangian, $L = L_{\mathrm{YM}} + L_{\mathrm{f}}$. It has the standard form except for the N_g factor in front of the gauge field kinetic terms that comes from the trace over generations. The analogous term in L_{f} give the sum over generations. We know that there are only three light generations of fermions, but is that all? We should count all generations in L! This means that the coefficient in front of the $F_{\mu\nu}F^{\mu\nu}$ terms should depend on N_g and, in fact, give us information about the total numbers of generations because it is absent from the fermionic part! This is not true. The orthodox normalization is correct. We should normalize the Diximier trace in (10) so that the coefficient N_g disappears. The simplest and most natural solution is to normalize Tr so that $\mathrm{Tr}\, \mathrm{Id}_{N_g} = 1$ (Chamsedine *et al.*, 1992*a,b*). This ensures also that Tr_ω is always finite. There is a natural inner product on the algebra of complex square matrices given by $\mathrm{Tr}(AB^\dagger)$. If one applies the Cauchy–Schwarz inequality to this inner product, one gets

$$\mathrm{Tr}(MM^\dagger)^2 \le (\mathrm{Tr}\, MM^\dagger)^2 \tag{16}$$

We cannot ensure the correct sign of the Higgs mass term without the above normalization. The normalization of the trace Tr leads to

$$\mathrm{Tr}(MM^\dagger)^2 \le N_g (\mathrm{Tr}\, MM^\dagger)^2 \tag{17}$$

This means that for large N_g the coefficient $K = \mathrm{Tr}(MM^\dagger)^2 - (\mathrm{Tr}\, MM^\dagger)^2$ may be very large. In fact, it is possible that $K \to \infty$ if the number of heavy generations is unlimited. This forces the condition $HH^\dagger = 1$ in the Lagrangian and removes the Higgs particle from the spectrum! If we are going to interpret the Yukawa coupling in the standard way, then we are not allowed to arbitrarily rescale the Higgs field and the limiting case leads to

$$m_{\rm H} = \left[2 \frac{\mathrm{Tr}(MM^\dagger)^2 - (\mathrm{Tr}\, MM^\dagger)^2}{\mathrm{Tr}\, MM^\dagger} \right]^{1/2} \to \infty \tag{18}$$

as should be expected. The fermionic masses are in such a (nonlinear) model by means of Yukawa couplings in a way analogous to that of the standard model (Cosalbuoni *et al.*, 1987; Cvetic and Kögerler, 1989, 1991; Bönish and Kögerler, 1992). The fermionic part of the Lagrangian given by equation (15) has the required form!

Another interesting possibility is to consider a "more symmetric" version containing two $SU(2)$ factors. Then, in order to have adjoint Higgs representations, one has to extend the spacetime by two points for each $SU(2)$ factor and identify the two copies (Chamsedine *et al.*, 1992, 1993*a*). If the mechanism suggested above really works, one gets a model that predicts several interacting facts (Cvetic and Kögerler, 1989, 1991; Bönish and Kögerler, 1992; Bönish *et al.*, 1992). For example, the two- and three-vector-boson-production process in e^+e^- collisions at $\sqrt{s} = 500$ GeV (NLC) will give precise bounds for the parameters (Bönish *et al.*, 1992).

Let us conclude by saying that the BESS mechanism is a necessary consequence of the noncommutative version of the standard model if there are many heavy generations. Such models are discrete counterparts of the CP^n sigma model obtained in the Kaluza–Klein program (Yoon, 1992) but are far more realistic: they predict interesting, experimentally verifiable facts. If we try to preserve the standard interpretation of the mass scale of the model (Connes, 1990; Connes and Lott, 1990), then our case corresponds to infinitesimal distance between copies of ordinary four-dimensional spacetime.

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